SIMULTANEOUS APPROXIMATION OF PERIODIC CONTINUOUS FUNCTIONS AND THEIR DERIVATIVES(1)

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ABSTRACT

Let $R_n(f; x)$ be a trigonometric polynomial of order n satisfying Eqs. (1.1) and (1.2). The object ofthis note is to obtain sufficient conditions in order that the pth derivative of $R_n(f, x)$ converges uniformly to $f^{(p)}(x)$ on the real line. The sufficient conditions turns out to be $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 0$ with the restrictions of Eq. (1.3).

1. In our earlier work [6] we obtained the explicit form of the trigonometric polynomial $R_n(x)$ of order n and established their uniqueness in the $(0, M)$ case, that is, when $(x_{k,n} = 2k\pi/n)$

$$R_n(x_{k,n}) = \alpha_{k,n}, \quad R_n^{(M)}(x_{k,n}) = \beta_{k,n}, \qquad k = 0, 1, 2, \cdots, n-1, \tag{1.1}$$

are prescribed, M being a fixed positive integer $\geqq 1$. Let $f(x)$ be a given periodic continuous function then we proved (see Theorem 2 and Theorem 3 of [6]) that for M odd the sequence of interpolatory polynomials satisfying (1.1) with $\alpha_{k,n} = f(x_{k,n})$ and $\beta_{k,n} = o(n^M/\log n)$ converges uniformly to $f(x)$ on the real axis but for M even we require $\beta_{k,n} = o(n^{M-1})$ and $f(x)$ satisfying a Zygmund condition. For $M = 1$ the polynomials have been dealt with by D. Jackson [see Zygmund Vol. 2 [8]] and when $M = 2$ the case has been treated by O. Kis [5]. It is also important to remark that we took $R_n(x)$ satisfying Eq. (1.1) having the form

$$R_n(x) = c_0 + \sum_{k=1}^{n-1} (c_k \cos kx + d_n \sin kx) + \begin{cases} d_n \sin nx, & M\text{—odd} \\ c_n \cos nx, & M\text{—even} \end{cases} \tag{1.2}$$

Let $f(x)$ be p-times a continuously differentiable periodic function. Now onwards we will assume throughout this paper M to be a fixed positive odd integer.

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(1) The author acknowledges financial support for this work from the University of Alberta, Post Doctoral Fellowship 1966–67.

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The main object of this note is to obtain sufficient conditions in order that the pth derivative of $R_n(f;x)$ converges uniformly to $f^{(p)}(x)$ on the real line. The sufficient conditions turns out to be $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$, $\alpha > o$ with certain restrictions on β_{kn}. There are many results analogous to this problem. For example it is well known that Bernstein polynomials provide simultaneous approximation of the function and their derivatives [see [2] p. 112]. Another recent important contribution of simultaneous approximation of functions and derivatives, by means of interpolatory polynomials, is due to Professor Geza Freud [3]. He investigated the sufficient conditions under which the pth differentiated sequence of Lagrange interpolation polynomials (associated with the fundamental point system $\{x_{kn}\}$ which are the zeros of certain orthogonal polynomials) converges uniformly to $f^{(p)}(x)$ in $[a,b] \subset (-1+1)$. As he proved in Theorem 1 of his paper, the sufficient conditions are $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$, $\alpha > \frac{1}{2}$.

Remark. After the paper was written and sent for publication it was very rightly pointed out by the referee my unawareness of the extremely interesting work of the late J. Czipszer and Professor G. Freud which had appeared in 1958, Acta Math. (99) 33–51. This work is closely connected with my Theorem 1.2. They attacked very successfully the general problem namely given the estimate,

$$\max |f(x) - P_n(x)| \leqq \varepsilon$$

what could one say about

$$\max |f^{(k)}(x) - P_n^{(k)}(x)|$$

provided $f^{(k)}(x)$ exists periodic, of period 2π, and is continuous? In fact by applying the quoted authors' result one needs only to find the estimate for (1.5) for the case $s = 0$. Therefore, following their theorem, one needs to prove Lemma 3,1 and Lemma 3.2 only for $s = 0$. But the general proof of these lemmas does not present us with any difficulty once we use the idea of positivity of Fejer Kernel and express the fundamental polynomials in the form of Fejer Kernel as given in (3.3). Moreover general results as stated in Lemmas 3.1 and 3.2 will be useful also for later work in application to different problems. The conclusion (1.5) of Theorem 1.2 by applying both approaches leads us to the same result as far as the order of n is concerned.

More precisely we shall prove

Theorem 1.1. *Let $f(x)$ be a continuous periodic function and $f^{(p)}(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 0$ and let*

$$|\beta_{kn}| = o\left(\frac{n^{M-p}}{\log n}\right) \qquad k = 0, 1, 2 \cdots, n-1. \tag{1.3}$$

Then the p^{th} derivative of $R_n(f;x)$ will converge uniformly to $f^{(p)}(x)$ on the real line.

THEOREM 1.2. *Let $f(x)$ be a periodic continuous function having continuous b derivatives in* $(0, 2\pi)$ *and* $f^{(q)}(x) \in \operatorname{Lip} \alpha$, $0 < \alpha < 1$, *and let*

$$\beta_{k,n} = 0(n^{M-q-\alpha}) \qquad k = 0,1,2,\cdots,n-1 \quad . \tag{1.4}$$

Then

$$\left|(R_n^{(s)}(x) - f^{(s)}(x)\right| = 0\left(\frac{\log n}{n^{q-s+\alpha}}\right) \qquad s = 0,1,\cdots,q. \tag{1.5}$$

It is interesting to remark that if we denote by $Q_n(x)$ the polynomial of best approximation corresponding to a given periodic function having qth derivative belonging to $\operatorname{Lip}\alpha$, $0 < \alpha < 1$ it is well known that [1], [9]

$$\left|Q_n^s(x) - f^{(s)}(x)\right| = 0\left(\frac{1}{n^{q-s+\alpha}}\right) \qquad s = 0,1,\cdots,q.$$

Thus the trigonometrical polynomial $R_n(x)$ obtained from $(0, M)$ interpolation does not yield as good approximation to $f(x)$ as compared to $Q_n(x)$ although it is not very far from it. Further it will be clear from the proof the main role played by the Fejer type Kernel.

2. **Preliminaries.** The trigonometric polynomials $R_n(x)$ of the form (1.2) satisfying (1.1) is given by

$$R_n(x) = \sum_{k=0}^{n-1} \alpha_{k,n} F(x - x_{k,n}) + \sum_{k=0}^{n-1} \beta_{k,n} G(x - x_{k,n}) \tag{2.1}$$

where

$$F(x) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{n-1} \frac{(n-j)^M \cos jx}{(n-j)^M + j^M}\right] \tag{2.2}$$

and

$$G(x) = (-1)^{M-1/2}\left[\frac{2}{n}\sum_{j=1}^{n-1} \frac{\sin jx}{(n-j)^M + j^M} + \frac{\sin nx}{n^{M+1}}\right] \tag{2.3}$$

We shall need the following known results.
Let us denote

$$t_{j,k} \equiv t_{j,k}(x - x_{k,n}) = 1 + \frac{2}{j}\sum_{i=1}^{j-1} (j-i) \cos i(x - x_{k,n}) \tag{2.4}$$

which is known as Fejer Kernel [Zygmund Vol. II page 21 [8]] so that

$$\sum_{k=0}^{n-1} t_{j,k} = n, \qquad j t_{jk} = \left[\frac{\sin j\dfrac{(x - x_{k,n})}{2}}{\sin \dfrac{x - x_{k,n}}{2}}\right]^2 \tag{2.5}$$

It is easy to see that

$$(2.6) \qquad (j+1)t_{j+1,k} - 2j\,t_{j,k} + (j-1)\,t_{j-1,k} = 2\cos j(x-x_k)$$

From (2.4) on differentiating $t_{j,k}$ $2l$ times with respect to x and using a similar relation as in (2.6) we get

$$(2.7) \qquad j\,t_{j,k}^{(2l)} = (-1)^{l}\left[\sum_{i=1}^{j-1} i\,b_{i,j}t_{i,k} + j(j-1)^{2l}t_{j,k}\right],$$

where we have denoted

$$(2.8) \qquad b_{i,j} = (j-i+1)(i-1)^{2l} - 2(j-i)i^{2l} + (j-i-1)(i+1)^{2l}.$$

Looking $b_{i,j}$ as the second difference of $(j-i)\,i^{2l}$ we get

$$(2.9) \qquad |b_{i,j}| \leqq 8l^2 j^{2l-1}.$$

From (2.5) and (2.7) using (2.9) we may conclude that

$$(2.10) \qquad \sum_{k=0}^{n-1} |t_{j,k}^{(2l)}| \leqq (8l^2+1)\,n\,j^{2l}.$$

We will also need the estimate of

$$(2.11) \qquad c_{n,j} = p(j-1) - 2p(j) + p(j+1),\; p(j) = \frac{(n-j)^M}{(n-j)^M + j^M}$$

as given in [6] page 349

$$(2.12) \qquad |c_{n,j}| \leqq \frac{M(M^2-1)\,2^{3M-1}}{n^2}.$$

From (2.5) and the estimate

$$(2.13) \qquad \left|\left[\frac{\sin j\dfrac{(x-x_{k,n})}{2}}{\sin\dfrac{(x-x_{k,n})}{2}}\right]'\right| \leqq j^2$$

(where dash denotes differentiation with respect to x) we have

$$|t'_{j,k}| \leqq 2j\left|\frac{\sin j\dfrac{(x-x_{k,n})}{2}}{\sin\dfrac{(x-x_{k,n})}{2}}\right|$$

Now using the result of D. Jackson [4] page 120 we get

$$(2.14) \qquad \sum_{k=0}^{n-1} |t'_{j,k}| \leqq 2c_1\,j\,n\log n.$$

3. With the above results stated in §2 we shall now obtain estimates for the derivatives of the fundamental polynomials of $(0, M)$ interpolation.

LEMMA 3.1. *We have*

$$\sum_{k=0}^{n-1} \left| F^{(2l)}(x - x_{k,n}) \right| \leqq c_1(M, l) n^{2l}, \tag{3.1}$$

and

$$\sum_{k=0}^{n-1} \left| F^{(2l+1)}(x - x_{k,n}) \right| \leqq c_2(M, l) n^{2l+1} \log n. \tag{3.2}$$

Proof. From (2.2), (2.6) and (2.11) we get

$$F(x - x_{k,n}) = \frac{1}{n} \sum_{j=1}^{n-1} j c_{n,j} t_{j,k} + \frac{t_{n,k}}{(n-1)^M + 1^M}. \tag{3.3}$$

Therefore

$$F^{(2l)}(x - x_{k,n}) = \frac{1}{n} \sum_{j=0}^{n-1} j c_{n,j} t_{j,k}^{(2l)} + \frac{t_{n,k}^{(2l)}}{(n-1)^M + 1^M},$$

so we get

$$\sum_{k=o}^{n-1} F^{(2l)}(x - x_{kn}) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=1}^{n-1} j \left| c_{n,k} \, (t_{j,k}^{(2l)}) \right| + \frac{1}{(n-1)^M + 1} \sum_{k=0}^{n-1} \left| t_{n,k}^{(2l)} \right|.$$

Now changing the order of summation which is justified for both summations are finite we have

$$\sum_{k=0}^{n-1} \left| F^{(2l)}(x - x_{k,n}) \right| \leqq \frac{1}{n} \sum_{j=1}^{n-1} j \left| c_{n,j} \right| \sum_{k=0}^{n-1} \left| t_{j,k}^{(2l)} \right| + \frac{1}{(n-1)^M + 1^M} \sum_{k=0}^{n-1} \left| t_{n,k}^{(2l)} \right|.$$

Now by using (2.10) and (2.12) a simple computation gives (3.1). In order to obtain (3.2) we observe that by (2.7)

$$j t_{j,k}^{(2l+1)} = (-1)^l \left[\sum_{i=1}^{j-1} i \, b_{i,j} t'_{i,k} + j(j-1)^{2l} t'_{j,k} \right]. \tag{3.4}$$

Now using (2.14) and proceeding in the same way as above we get (3.2).

LEMMA 3.2 *We have*

$$\sum_{k=0}^{n-1} \left| G^{(p)}(x - x_{k,n}) \right| \leqq c_3(p, M) n^{p-M} \log n\,. \tag{3.5}$$

Proof. From (2.3) we have

$$\left| G^{(2l-1)}(x - x_{k,n}) \right| = \left| \frac{2}{n} \sum_{j=1}^{n-1} \frac{j^{2l-1}}{(n-j)^M + j^M} \cos j(x - x_{k,n}) + n^{2l-M-2} \cos n(x - x_{k,n}) \right|. \tag{3.6}$$

Now using (2.6) and putting $q_{j,n} = \dfrac{j^{2l-1}}{(n-j)^M + j^M}$ we get

$$\left|G^{(2l-1)}(x - x_{k,n})\right| \leqq \tag{3.7}$$

$$\left|\frac{2}{n} \sum_{j=1}^{n-1} (q_{j+1,n} - 2q_{j,n} + q_{j-1,n}) j\, t_{j,k} + (q_{n-1,n}\, n\, t_{n,k} - q_{n,n}(n-1)\, t_{n-1,k})\right| + n^{2l-M-2}$$

A simple computation gives

$$\left|q_{j+1,n} - 2q_{j,n} + q_{j-1,n}\right| \leqq c_3 j^{2l-3-M}\,, \tag{3.8}$$

and

$$\left|q_{n,n} - q_{n-1,n}\right| \leqq c_2 n^{2l-2-M}\,. \tag{3.9}$$

But from (2.4) we get

$$n\, t_{n,k} - (n-1)\, t_{n-1,k} = 1 + 2 \sum_{i=1}^{n-1} \cos i(x - x_{k,n})\,. \tag{3.10}$$

Therefore

$$n\, q_{n-1,n} t_{n,k} - (n-1)\, q_{n-1,n} t_{n-1,k}$$

$$= (q_{n-1,n} - q_{n,n})\,(n-1)\, t_{n-1} + q_{n-1,n}\Big(1 + 2 \sum_{i=1}^{n-1} \cos i(x - x_{k,n})\Big)$$

Thus (3.7) now becomes

$$\left|G^{(2l-1)}(x - x_{k,n})\right| \leqq \frac{2}{n}\left[\sum_{j=1}^{n-1} j^{2l-2-M} c_1\, t_{j,k} + c_2\, n^{2l-1-M} t_{n-1,k}\right.$$

$$\left. + \; n^{2l-1-M} \left| \frac{\sin \frac{(2n-1)}{2}(x - x_{k,n})}{\sin \frac{x - x_{k,n}}{2}} \right| \right] + n^{2l-M-2}$$

Now using the known result

$$\sum_{k=0}^{n-1} \left| \frac{\sin \frac{(2n-1)}{2}(x - x_{k,n})}{\sin \frac{x - x_{k,n}}{2}} \right| = c_4\, n \log n$$

[see D. Jackson [4] p. 120], we get by using also formula (1.5) the result stated in (3.6) for $p = 2l - 1$. To prove (3.6) for $p = 2l$ we differentiate once with respect to x equation (3.7) and use (2.14).

LEMMA 3.3. *Let $f(x)$ be a periodic continuous function and $f^{(p)}(x) \in \text{Lip}\ \alpha$, $\alpha > 0$. Then there exists a sequence of trigonometric polynomials $T_n(x)$ of order $n-1$ such that*

$$|f(x) - T_n(x)| = 0\left(\frac{1}{n^{p+\alpha}}\right), \tag{3.12}$$

$$|f^{(p)}(x) - T_n^{(p)}(x)| = 0\left(\frac{1}{n^{\alpha}}\right), \tag{3.13}$$

$$|T_n^{(l)}(x)| = 0(n^{l-p-\alpha}) \qquad l = 1, 2, \cdots. \tag{3.14}$$

The existence of a sequence satisfying (3.12) and (3.13) is well known (cf. [7,9]). The proof for (3.14) is simi!ar to that of Lemma 7 of [6].

4. **Proof of Theorem 1.1.** From the uniqueness theorem (see theorem 1 of [6]). we have

$$T_n(x) = \sum_{k=0}^{n-1} T_n(x_{k,n}) F(x - x_{k,n}) + \sum_{k=0}^{n-1} T_n^{(M)}(x_{k,n}) G(x - x_{k,n})). \tag{4.1}$$

But we know that

$$R_n(x) = \sum_{k=0}^{n-1} f(x_{k,n}) F(x - x_{k,n}) + \sum_{k=0}^{n-1} \beta_{k,n} G(x - x_{k,n}). \tag{4.2}$$

Since

$$R_n^{(p)}(x) - f^{(p)}(x) = R_n^{(p)}(x) - T_n^{(p)}(x) + T_n^{(p)}(x) - f^{(p)}(x),$$

we get using (3.13)

$$|R_n^{(p)}(x) - f^{(p)}(x)| = |R_n^{(p)}(x) - T_n^{(p)}(x)| + 0\left(\frac{1}{n^{\alpha}}\right). \tag{4.3}$$

Using (4.1), (4.2), (3.12), Lemma 3.1 and Lemma 3.2 we get

$$\begin{aligned} |T_n^{(p)}(x) - R_n^{(p)}(x)| \leq & \sum_{k=0}^{n-1} 0\left(\frac{1}{n^{p+\alpha}}\right) |F^{(p)}(x - x_{k,n})| \\ & + \sum_{k=0}^{n-1} 0(n^{M-p-\alpha}) |G^{(p)}(x - x_{k,n})| \\ & + \sum_{k=0}^{n-1} |\beta_{k,n}| |G^{(p)}(x - x_{k,n})|, \end{aligned}$$

using (1.3) we get $|T_n^{(p)}(x) - R_n^{(p)}(x)| = o(1)$. Proof of Theorem 1.2 follows from Lemmas 3.1 – 3.3. We omit the details.

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